

YOUNG DIAGRAMS AND N -SOLITON SOLUTIONS OF THE KP EQUATION

YUJI KODAMA

ABSTRACT. We consider N -soliton solutions of the KP equation,

$$(-4u_t + u_{xxx} + 6uu_x)_x + 3u_{yy} = 0.$$

An N -soliton solution is a solution $u(x, y, t)$ which has the same set of N line soliton solutions in both asymptotics $y \rightarrow \infty$ and $y \rightarrow -\infty$. The N -soliton solutions include all possible resonant interactions among those line solitons. We then classify those N -soliton solutions by defining a pair of N -numbers $(\mathbf{n}^+, \mathbf{n}^-)$ with $\mathbf{n}^\pm = (n_1^\pm, \dots, n_N^\pm)$, $n_j^\pm \in \{1, \dots, 2N\}$, which labels N line solitons in the solution. The classification is related to the Schubert decomposition of the Grassmann manifolds $\text{Gr}(N, 2N)$, where the solution of the KP equation is defined as a torus orbit. Then the interaction pattern of N -soliton solution can be described by the pair of Young diagrams associated with $(\mathbf{n}^+, \mathbf{n}^-)$. We also show that N -soliton solutions of the KdV equation obtained by the constraint $\partial u / \partial y = 0$ cannot have resonant interaction.

1. INTRODUCTION

In this paper, we consider a family of exact solutions of the KP equation,

$$\frac{\partial}{\partial x} \left(-4 \frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + 6u \frac{\partial u}{\partial x} \right) + 3 \frac{\partial^2 u}{\partial y^2} = 0.$$

The solution $u(x, y, t)$ is obtained from the τ -function $\tau(x, y, t)$ as

$$u(x, y, t) = 2 \frac{\partial^2}{\partial x^2} \log \tau(x, y, t).$$

It is well known that some solutions can be obtained by the Wronskian form (for example see [2]),

$$(1.1) \quad \tau = \text{Wr}(f_1, \dots, f_N) := \begin{vmatrix} f_1^{(0)} & \cdots & f_N^{(0)} \\ \vdots & \ddots & \vdots \\ f_1^{(N-1)} & \cdots & f_N^{(N-1)} \end{vmatrix},$$

where $f_i^{(n)} = \partial^n f_i / \partial x^n$, and $\{f_i(x, y, t) \mid i = 1, \dots, N\}$ is a linearly independent set of N solutions of the equations,

$$\frac{\partial f_i}{\partial y} = \frac{\partial^2 f_i}{\partial x^2} \quad \frac{\partial f_i}{\partial t} = \frac{\partial^3 f_i}{\partial x^3}.$$

Key words and phrases. the KP equation, N -soliton solution, tau-function, Grassmannian, Schubert decomposition, Young diagram.

Throughout this paper, we assume $f_i(x, y, t)$ to have the form,

$$(1.2) \quad f_i = \sum_{j=1}^M a_{ij} e^{\theta_j}, \quad \text{for } i = 1, \dots, N, \quad \text{and } M > N,$$

with some constants a_{ij} which define the $N \times M$ matrix $A_{(N,M)} := (a_{ij})$, and the phase functions θ_j are given by

$$(1.3) \quad \theta_j(x, y, t) = -k_j x + k_j^2 y - k_j^3 t + \theta_j^0 \quad \text{for } j = 1 \dots, M.$$

Here k_j and θ_j^0 are arbitrary constants, and throughout this paper we assume k_j being ordered as

$$k_1 < k_2 < \dots < k_M.$$

Then choosing particular forms for the matrix $A_{(N,M)}$, one can obtain several exact solutions of the KP equation. As the simplest case with $N = 1$ and $M = 2$, i.e. $\tau = f_1 = a_{11}e^{\theta_1} + a_{12}e^{\theta_2}$ with $a_{11}a_{12} > 0$, we have 1-soliton solution,

$$(1.4) \quad u = 2 \frac{\partial^2}{\partial x^2} \log \tau = \frac{1}{2} (k_1 - k_2)^2 \operatorname{sech}^2 \frac{1}{2} (\theta_1 - \theta_2).$$

Here we assumed $a_{11}, a_{12} > 0$ and absorbed them to the constants θ_j^0 , so that we have $a_{11} = a_{12} = 1$. Note here that if $a_{11}a_{12} < 0$, then the τ -function has zeros and the solution blows up at some points in the x - y plane. In the x - y plane, 1-soliton solution describes a plane wave $u = \Phi(k_x x + k_y y - \omega t)$ having the wavenumber vector $\mathbf{k} = (k_x, k_y)$ and the frequency ω ,

$$\mathbf{k} = (-k_1 + k_2, k_1^2 - k_2^2) \quad \omega = k_1^3 - k_2^3.$$

Here (\mathbf{k}, ω) satisfies the dispersion relation, $4\omega k_x + k_x^4 + 3k_y^2 = 0$. We refer to the 1-soliton solution as a line soliton, which is expressed by a line $\theta_1 = \theta_2$ in the x - y plane for a fixed t . Then the slope of the line $c = dx/dy = -k_y/k_x$ represents the velocity of the line soliton in the x -direction with respect to y ; that is, $c = 0$ indicates the direction of the positive y -axis. As in the paper [1], we also call a line soliton as $[i, j]$ -soliton, if the soliton is parametrized by the pair (k_i, k_j) . In the case of (1.4), the soliton is $[1, 2]$ -soliton with the velocity $c = k_1 + k_2$. Then the ordinary N -soliton solution consisting of N line solitons, $[2j - 1, 2j]$ -soliton with $j = 1, \dots, N$, is obtained from the following matrix $A_{(N,M)}$ with $M = 2N$ [5],

$$A_{(N,2N)} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \cdots & \cdots & \cdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{pmatrix},$$

Note here that all nonzero entries are normalized to be one by choosing appropriate constants θ_j^0 , and any $N \times N$ minor of the matrix $A_{(N,2N)}$ is non-negative. The fact that all $N \times N$ minors are non-negative (or non-positive) is sufficient for the solution to be non singular (see below).

Also in the paper [1], we found that if all the minors are nonzero and have the same sign, the solution u gives an $(M - N, N)$ -soliton solution consisting of $M - N$ incoming line solitons as $y \rightarrow -\infty$ and N outgoing line solitons as $y \rightarrow \infty$. The set of the τ -functions with different size N then provide the solution of the Toda lattice. In particular, if $M = 2N$, we have N -soliton solution in the sense that the solution has the same set of N line solitons in both asymptotics for $y \rightarrow \pm\infty$ (i.e.

the sets of incoming and outgoing solitons are the same). However as mentioned in [1], this N -soliton solution is different from the ordinary N -soliton solution, and the interaction of any pair of line solitons is in resonance. In this case, the matrix $A_{(N,2N)}$ can be written in the following row reduced echelon form (RREF),

$$A_{(N,2N)} = \begin{pmatrix} 1 & 0 & \cdots & 0 & * & * & \cdots & * \\ 0 & 1 & \cdots & 0 & * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & * & * & \cdots & * \end{pmatrix}$$

where the entries marked by “*” are chosen so that all the $N \times N$ minors are nonzero and positive (i.e. all the *’s of the last row should be all positive, and those in the second row from the bottom are negative and so on). Note here that those *’s are all nonzero. Then the line solitons in N -soliton solution of the Toda lattice hierarchy are given by $[j, N + j]$ -soliton for $j = 1, \dots, N$.

We then expect that the general case of N -soliton solutions has a mixed pattern consisting of resonant and non-resonant interactions among those line solitons. In this paper, we classify all the possible N -soliton solutions obtained by the τ -function (1.1) with f_j ’s in (1.2) and $M = 2N$. The classification is to determine all possible patterns of interactions, and it is done by constructing a specific form of the coefficient matrix $A_N := A_{(N,2N)}$ in (1.2) for each class of N -soliton solutions. It turns out that a complete classification of N -soliton solutions can be obtained by the complementary pair of ordered sets of N -numbers, $\mathbf{n}^+ = (n_1^+, \dots, n_N^+)$ and $\mathbf{n}^- = (n_1^-, \dots, n_N^-)$, which are related to the dominant exponentials in the τ -function for $x \rightarrow \pm\infty$ and each soliton is given by $[n_j^+, n_j^-]$ -soliton. Notice that with the set $\{1, \dots, 2N\}$, there are $(2N - 1)!!$ number of different sets of N pairs. We then claim that each set of N pairs corresponds to an N -soliton solution, and provide an explicit way to construct all those N -soliton solutions.

It is also well-known that a solution of the KP hierarchy is a $\text{GL}(\infty)$ -orbit on infinite dimensional Grassmannian (Sato Grassmannian) [9]. Then the N -soliton solution can be identified as the $2N$ -dimensional torus orbit on the Grassmannian $\text{Gr}(N, 2N)$, and the coefficient matrix A_N represents a point on $\text{Gr}(N, 2N)$. Since the Grassmannian has the Schubert cell decomposition, one can first classify the matrix A_N using the decomposition, i.e. identify the cell A_N belongs to. This classification corresponds to the set \mathbf{n}^+ , and the \mathbf{n}^- gives a further decomposition of the cells to classify the interaction patterns of N -soliton solutions. We then define a pair of Young diagrams (Y^+, Y^-) associated with the pair of the number sets $(\mathbf{n}^+, \mathbf{n}^-)$. In particular, the number of resonant vertices in the N -soliton solution parametrized by (Y^+, Y^-) is given by

$$\frac{N(N-1)}{2} - |Y^+| - |Y^-|,$$

where $|Y^\pm|$ denote the size (degree) of the diagrams, i.e. the number of boxes in the diagrams (Theorem 4.3).

The paper is organized as follows: In Section 2, we give a general basic structure of the τ -function, and present all types of 2-soliton solutions, which provide the building blocks for the general case of N -soliton solutions. Here we also introduce the pair of numbers $(\mathbf{n}^+, \mathbf{n}^-)$, and the Young diagrams (Y^+, Y^-) . In Section 3, we briefly introduce the Grassmannian $\text{Gr}(N, M)$ and the Schubert decomposition of

$\text{Gr}(N, M)$. We also identify N -soliton solution as an $2N$ -dimensional torus orbit on $\text{Gr}(N, 2N)$, and briefly mention that $\text{Gr}(N, 2N)$ contains all possible (m, n) -soliton solutions for $1 \leq m \leq N$ and $1 \leq n \leq N$, which are distinguished by the Schubert decomposition. Here (m, n) -soliton is the solution consisting of m incoming solitons for $y \rightarrow -\infty$ and n outgoing solitons for $y \rightarrow \infty$ (see [1]). In Section 4, we describe the structure of the coefficient matrix A_N for each N -soliton solution by prescribing an explicit construction of the matrix A_N . We here define N -soliton condition on the matrix A_N which determines local structures based on the types of 2-soliton interactions in the N -soliton solution. Finally, in Section 5, we discuss N -soliton solutions of the KdV equation, and show that the KdV N -soliton solution cannot have resonant interactions.

2. BASIC STRUCTURE OF THE τ -FUNCTION AND 2-SOLITON SOLUTIONS

Let us start with the following Lemma which shows the basic structure of the τ -function:

Lemma 2.1. *The τ -function (1.1) with f_j 's given in (1.2) can be expanded as a sum of exponential functions,*

$$\tau = \sum_{1 \leq i_1 < \dots < i_N \leq M} \xi(i_1, \dots, i_N) \prod_{1 \leq j < l \leq N} (k_{i_j} - k_{i_l}) \exp \left(\sum_{j=1}^N \theta_{i_j} \right),$$

where $\xi(i_1, \dots, i_N)$ is the $N \times N$ minor given by the i_j -th columns with $j = 1, \dots, N$ in the matrix $A_{(N,M)} = (a_{ij})$ of (1.2),

$$\xi(i_1, \dots, i_N) := \begin{vmatrix} a_{1,i_1} & \cdots & a_{1,i_N} \\ \vdots & \ddots & \vdots \\ a_{N,i_1} & \cdots & a_{N,i_N} \end{vmatrix}.$$

Proof. Apply the Binet-Cauchy theorem for the expression (see for example [3]),

$$\tau = \left| \begin{pmatrix} E_1^{(0)} & E_2^{(0)} & \cdots & E_M^{(0)} \\ E_1^{(1)} & E_2^{(1)} & \cdots & E_M^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ E_1^{(N-1)} & E_2^{(N-1)} & \cdots & E_M^{(N-1)} \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{N1} \\ a_{12} & a_{22} & \cdots & a_{N2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1M} & a_{2M} & \cdots & a_{NM} \end{pmatrix} \right|,$$

where $E_j^{(n)} = (-k_j)^n E_j$ with $E_j = e^{\theta_j}$. \square

From this Lemma, it is clear that if all the minors $\xi(i_1, \dots, i_N)$ are non-negative (or non-positive), then the τ -function is sign definite and the corresponding solution u is nonsingular (recall also the order $k_1 < \dots < k_M$). This is the first restriction on the matrix $A_{(N,M)}$ for N -soliton solution, and we later impose that all the $N \times N$ minors are non-negative after making $A_{(N,M)}$ to be in the row reduced echelon form.

We now consider the case of N -soliton solutions, that is, we take $M = 2N$. With the ordering of the numbers k_j 's, i.e. $k_1 < k_2 < \dots < k_{2N}$, one can find the

asymptotic behaviour of the τ -function for $x \rightarrow \pm\infty$: For this purpose, we first define the function w given by

$$w(x, y, t) = -\frac{\partial}{\partial x} \log \tau(x, y, t).$$

Suppose that w has the asymptotic form with some numbers $\{n_1^\pm, \dots, n_N^\pm\} \subset \{1, \dots, 2N\}$,

$$(2.1) \quad w \longrightarrow \begin{cases} \sum_{j=1}^N k_{n_j^+} & \text{as } x \rightarrow \infty, \\ \sum_{j=1}^N k_{n_j^-} & \text{as } x \rightarrow -\infty. \end{cases}$$

It is clear that $n_i^+ \neq n_j^+$ and $n_i^- \neq n_j^-$ if $i \neq j$. Then with the sets of N numbers n_j^\pm 's in the asymptotics (2.1), we define the following two N -vectors with the entries from the set $\{1, 2, \dots, 2N\}$:

Definition 2.1. A pair $(\mathbf{n}^+, \mathbf{n}^-)$ of N -vectors are defined by

$$\begin{cases} \mathbf{n}^+ = (n_1^+, n_2^+, \dots, n_N^+) & \text{with } 1 = n_1^+ < n_2^+ < \dots < n_N^+ < 2N, \\ \mathbf{n}^- = (n_1^-, n_2^-, \dots, n_N^-) & \text{with } n_j^- > n_j^+. \end{cases}$$

The ordering in the set \mathbf{n}^+ is just a convenience, and once we made the ordering in \mathbf{n}^+ , we do not assume a further ordering for other set \mathbf{n}^- . We assume that those sets are complementary in the set $\{1, 2, \dots, 2N\}$. As will be explained below, this assumption is necessary to have an N -soliton solution, that is, the solution has the same sets of N line solitons for both asymptotics $y \rightarrow \pm\infty$.

With those number sets \mathbf{n}^+ and \mathbf{n}^- , one can define the pair of Young diagrams (Y^+, Y^-) :

Definition 2.2. We define a pair (Y^+, Y^-) associated to the pair $(\mathbf{n}^+, \mathbf{n}^-)$:

- The Y^+ is a Young diagram represented by $(\ell_1^+, \dots, \ell_N^+)$ with $\ell_j^+ = n_j^+ - j$ where each ℓ_j^+ represents the number of boxes in a row counting from the bottom (note $\ell_j^+ \leq \ell_{j+1}^+$).
- The Y^- is a Young diagram associated to $(\ell_1^-, \dots, \ell_N^-)$, where ℓ_j^- is defined as

$$\ell_j^- = \left| \left\{ n_k^- \mid n_k^- < n_j^- \text{ for } k > j \right\} \right|.$$

Then ℓ_j^- represents the number of boxes in a row arranged in the increasing order from the bottom.

Notice that the Young diagram Y^+ is uniquely determined from \mathbf{n}^+ , but the diagram Y^- is not unique (note that $(\ell_1^-, \dots, \ell_N^-)$ is unique of course). However the number of boxes in Y^- gives the number of intersection vertices of N -soliton solution having a particular type of interaction (Theorem 4.3).

As a main goal of this paper, we will show in Section 4 that for each pair $(\mathbf{n}^+, \mathbf{n}^-)$ one can construct an N -soliton solution consisting of N line solitons, each of which is given by $[n_j^+, n_j^-]$ -soliton for $j = 1, \dots, N$. The total number of different N -soliton solutions is given by $(2N - 1)!!$ which is the number of different sets of pairs $\{(\mathbf{n}^+, \mathbf{n}^-)\}$. Namely, we construct an N -soliton solution which is labeled by $(\mathbf{n}^+, \mathbf{n}^-)$ as follows:

- 1) Consider a set of $2N$ numbers $\{1, \dots, 2N\}$, which are associated to the parameters $k_1 < \dots < k_{2N}$.
- 2) Choose N different pairs from $\{1, \dots, 2N\}$, and label each pair as $[n_j^+, n_j^-]$ so that the numbers n_j^\pm satisfy
 - i) $n_j^+ < n_j^-$ for $j = 1, \dots, N$,
 - ii) $1 = n_1^+ < n_2^+ < \dots < n_N^+$.
- 3) Define the pair $(\mathbf{n}^+, \mathbf{n}^-)$ with $\mathbf{n}^\pm = (n_1^\pm, \dots, n_N^\pm)$, and construct the corresponding matrix $A_N = A_{(N, 2N)}$.

Note here that the labeling is unique with a given choice of pairs, and the corresponding N -soliton solution consists of $[n_j^+, n_j^-]$ -solitons for $j = 1, \dots, N$. The item 3) will be given in Section 4.

In order to classify those N -soliton solutions, we first note:

Lemma 2.2. *For each given ordered set \mathbf{n}^+ , the number of choices of \mathbf{n}^- is given by*

$$m_{\mathbf{n}^+} = \prod_{j=1}^N (2j - n_j^+).$$

Proof. Since $n_N^- > n_N^+$, there are $(2N - n_N^+)$ numbers of possible choices of n_N^- . Having made a choice of n_N^- , one has $2(N-1) - n_{N-1}^+$ many choices for n_{N-1}^- . Now repeating this, the result is obvious. \square

This Lemma provides the number of different N -soliton solutions having the same \mathbf{n}^+ , that is, the functions $w = -(\partial/\partial x) \log \tau$ for those solutions have the same asymptotic values for $x \rightarrow \pm\infty$. As a corollary of this Lemma, we also have the following identity,

$$(2N - 1)!! = \sum_{\mathbf{n}^+} m_{\mathbf{n}^+}.$$

We also have the following Lemma for the ordered set \mathbf{n}^+ :

Lemma 2.3. *Each number n_j^+ in the \mathbf{n}^+ is limited as*

$$n_j^+ \leq 2j - 1 \quad \text{for } j = 1, \dots, N.$$

Proof. Since $n_j^+ < n_j^-$, there are $N - j$ numbers of n_k^+ and $N - j + 1$ of n_k^- , which are larger than n_j^+ , that is, we have

$$(N - j) + (N - j + 1) \leq 2N - n_j^+.$$

This implies the Lemma. \square

In terms of the Young diagram Y^+ , we have

Corollary 2.1. *The maximum diagram for Y^+ , denoted as Y_{\max} , is the Young diagram associated to $\mathbf{n}^+ = (1, 3, \dots, 2N - 1)$, that is, Y_{\max} is the up side down staircase with the size,*

$$|Y_{\max}| = \frac{N(N-1)}{2}.$$

The maximum Young diagram for Y^- is also given by Y_{\max} , which now corresponds to the case $\mathbf{n}^- = (2N, 2N-1, \dots, N+1)$. Then the Young diagrams Y^\pm associated to N -soliton solutions are given by a subdiagram of Y_{\max} .

Note that $|Y_{\max}|$ is also the number of intersection vertices of N lines in a plane in the general position. We will then determine whether each point is of resonant or non-resonant.

In Figure 1, we illustrate the asymptotic stage of an N -soliton solution with N solitons marked by $[n_j^+, n_j^-]$ for $j = 1, \dots, N$. We note a duality of the values of w in the sense that if w takes $\sum_{j=1}^N k_{i_j}$, denoted as (i_1, \dots, i_N) in the Figure, then w also takes (j_1, \dots, j_N) which is the complementary set of (i_1, \dots, i_N) in $\{1, \dots, 2N\}$. In terms of the minors, this implies that if $\xi(i_1, \dots, i_N) \neq 0$ then $\xi(j_1, \dots, j_N) \neq 0$. The duality can be considered as a symmetry of the KP equation: In particular, if $\theta_j^0 = 0$ for all j , then N -soliton solution has the symmetry $(x, y, t) \leftrightarrow (-x, -y, -t)$. In this case, at $t = 0$ all the solitons intersect at the origin in the x - y plane (see Figure 3). Then each soliton can be moved by shifting θ_j^0 . With this symmetry, we define the duality of the τ -function:

Definition 2.3. We say that the τ -function in Lemma 2.1 with $M = 2N$ satisfies the duality, if for any complementary sets $\{i_1, \dots, i_N\}$ and $\{j_1, \dots, j_N\}$, the minors satisfy

$$\xi(i_1, \dots, i_N) = 0 \quad \text{if and only if} \quad \xi(j_1, \dots, j_N) = 0.$$

Before we discuss the general case with arbitrary N , we list up all possible 2-soliton solutions, and show that the τ -functions for those cases all satisfy the duality. As will be shown below, those solutions give the building blocks of N -soliton solutions. There are three cases of the solutions (see also [6]), which are labeled by the pair $(\mathbf{n}^+, \mathbf{n}^-)$ with $\mathbf{n}^\pm = (n_1^\pm, n_2^\pm)$. Two line solitons are labeled by $[n_1^+, n_1^-]$ and $[n_2^+, n_2^-]$:

- i) $\mathbf{n}^+ = (1, 3)$ and $\mathbf{n}^- = (2, 4)$: This corresponds to the ordinary 2-soliton solution, and the matrix A_2 which we denote as $A_2^{(O)}$, takes the form (after choosing θ_j^0 properly),

$$A_2^{(O)} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

which gives only 4 nonzero minors, $\xi(1, 3)$, $\xi(1, 4)$, $\xi(2, 3)$ and $\xi(2, 4)$. This shows the duality, i.e. $\xi(1, 2) = \xi(3, 4) = 0$. Those nonzero minors then label the four asymptotic regions separated by two line solitons: Denoting $(i, j) := k_i + k_j$, w takes $(1, 3)$ as $x \rightarrow \infty$ which is dual to $(2, 4)$ as $x \rightarrow -\infty$. Also w takes $(1, 4)$ as $y \rightarrow \infty$, which is dual to $(2, 3)$ as $y \rightarrow -\infty$. This also implies that there is no other transition of the w value, hence the interaction is not a resonant type.

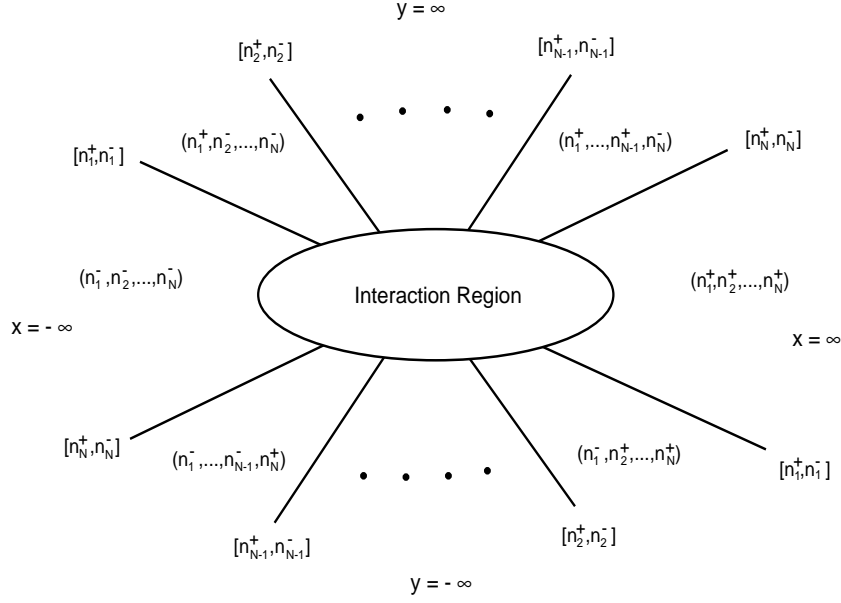


FIGURE 1. An N -soliton solution. The label $[n_j^+, n_j^-]$ indicates the $[n_j^+, n_j^-]$ -soliton. Here the soliton velocities $c_j := k_{n_j^+} + k_{n_j^-}$ are assumed to be ordered as $c_{j+1} > c_j$. Each asymptotic region is marked by the value of the function w there, i.e. (i_1, \dots, i_N) implies $w = \sum_{j=1}^N k_{i_j}$.

The Young diagram Y^+ in this case is the maximum one, i.e. $Y^+ = \square$. The diagram Y^- has no box, $Y^- = \emptyset$. The interaction of O-type is thus labeled by the Young diagram Y^+ . It is also useful to note that the intervals defined by the labels $[1, 2]$ and $[3, 4]$ of solitons have no overlaps. Non-overlapping property can be stated as $\xi(1, 2) = 0$ and its dual one $\xi(3, 4) = 0$.

- ii) $\mathbf{n}^+ = (1, 2)$ and $\mathbf{n}^- = (3, 4)$: This case corresponds to the 2-soliton solution of the Toda lattice [1], and the matrix A_2 , denoted as $A_2^{(T)}$, has the structure,

$$A_2^{(T)} = \begin{pmatrix} 1 & 0 & - & - \\ 0 & 1 & + & + \end{pmatrix},$$

where “+”, “-” shows the signs of the entries (also nonzero). An explicit example of $A_2^{(T)}$ is

$$A_2^{(T)} = \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

and the corresponding 2-soliton solution is illustrated as T-type in Figure 2. Following the arguments in [1], one can easily show that two solitons are given by $[1, 3]$ - and $[2, 4]$ -solitons, and the interaction is in resonance. The main point in this example is that all the 2×2 minors of $A_2^{(T)}$ are nonzero, and this results resonance interaction. Non-vanishing condition

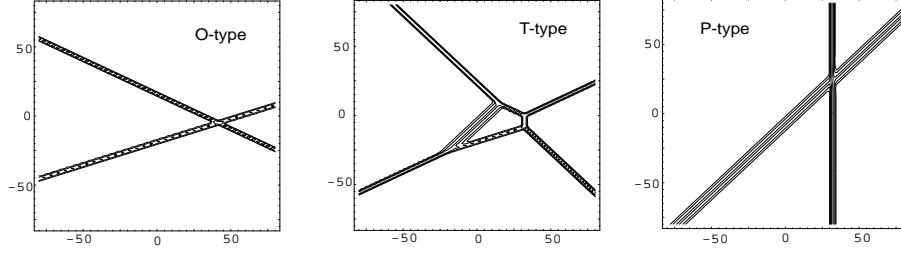


FIGURE 2. 2-soliton solutions. Two line solitons in those figures are $[1, 2]$ - and $[3, 4]$ -solitons for O-type, $[1, 3]$ - and $[2, 4]$ -solitons for T-type, and $[1, 4]$ -and $[2, 3]$ -solitons for P-type. The parameters are chosen as $(k_1, k_2, k_3, k_4) = (-2, 0, 1, 2)$ and $\theta_j^0 = 0, \forall j$ for all types.

of all the minors is necessary to make the resonant interaction, and those minors describe the intermediate solitons which forms Y -shape resonant interaction (see [1] for more detail).

Again we notice the duality in the values of w in the asymptotics: w takes $(3, 4)$ for $x \rightarrow -\infty$ which is dual to $(1, 2)$ for $x \rightarrow \infty$. Also w takes $(1, 4)$ for $y \rightarrow \infty$, which is dual to $(2, 3)$ for $y \rightarrow -\infty$. One more duality appears in the inside of the resonant hole, that is, w takes $(1, 3)$ for $t > 0$, which is dual to $(2, 4)$ for $t < 0$ (or vice versa). Thus all six nonzero minors contribute to make the resonant interaction. The pair (Y^+, Y^-) of the Young diagrams is then given by (\emptyset, \emptyset) . Also note that the intervals defined by the labels of solitons, $[1, 3]$ and $[2, 4]$ have a partial overlap.

- iii) $\mathbf{n}^+ = (1, 2)$ and $\mathbf{n}^- = (4, 3)$: This case has been noted in [6], and the corresponding matrix A_2 , denoted as $A_2^{(P)}$ (" P " stands for *Physical*, see Remark 2.4), is given by (after choosing θ_j^0 properly),

$$A_2^{(P)} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix},$$

which gives again only four nonzero minors with $\xi(1, 2)$, $\xi(1, 3)$, $\xi(2, 4)$ and $\xi(3, 4)$. This implies also no resonance, and the solitons are given by $[1, 4]$ - and $[2, 3]$ -solitons. This solution is similar to the O-type, but the pair of the Young diagrams is different, and given by (\emptyset, \square) . This interaction of this type is labeled by the Young diagram Y^- . Also the intervals defined by the labels of solitons, $[1, 4]$ and $[2, 3]$, have complete overlap. This implies $\xi(2, 3) = 0$ and its dual $\xi(1, 4) = 0$.

One should note that we have the following order in the soliton velocities $c_{ij} := k_i + k_j$ among those solitons for given constants k_j with the order $k_1 < k_2 < k_3 < k_4$,

$$c_{12} < c_{13} < c_{14}, c_{23} < c_{24} < c_{34}.$$

Here note that c_{14} and c_{23} cannot have a definite order (even $c_{14} = c_{23}$ is possible). This ordering indicates that any 2-soliton solution can be classified as one of those three solitons. One should note here that the type for a given pair of solitons are completely determined by their labels $[n_j^+, n_j^-]$ for $j = 1, 2$. Namely the interaction

pattern is completely determined their asymptotics in $y \rightarrow \infty$, where two solitons are labeled.

Although the interaction patterns of the cases of O-type in ii) and P-type in iii) are the same, i.e. nonresonant, they describe orbits on different cells in terms of the Schubert decomposition of the Grassmannian, in which each cell is parametrized by a Young diagram. We will explain more details on this description in Section 3.

Remark 2.4. In [7], the KP equation was introduced to describe a transversal stability of the KdV soliton propagating along the x -axis. Then the order of the wavenumber in the y -direction is assumed to be much smaller than that in the x -direction. This implies that the wave described by the KP equation should be almost parallel to the y -axis for a better approximation within the physical setting. In this sense, the P-type solitons are more physical than other two types. The KdV two solitons are also obtained from the P-type solitons with $k_1 < k_2 < 0$ and $k_3 = -k_2$, $k_4 = -k_1$ (see Section 5).

3. GRASSMANNIAN $\text{Gr}(N, 2N)$ AND N -SOLITON SOLUTIONS

Here we briefly summarize the basics of the Grassmann manifold $\text{Gr}(N, M)$ in order to explain that the minors $\xi(i_1, \dots, i_N)$ in Lemma 2.1 provide a coordinate system for $\text{Gr}(N, M)$, the Plücker coordinates. Namely a solution given by the τ -function (1.1) can be marked by a point on $\text{Gr}(N, M)$. The purpose of this section is to identify an N -soliton solution as a $2N$ -dimensional torus orbit of a point on $\text{Gr}(N, 2N)$ marked by the minors $\xi(i_1, \dots, i_N)$. Then using the Schubert decomposition of the Grassmannian $\text{Gr}(N, 2N)$, we classify the orbits which represent the N -soliton solutions.

3.1. Grassmannian $\text{Gr}(N, M)$. A real Grassmannian $\text{Gr}(N, M)$ is the set of N -dimensional subspaces of \mathbb{R}^M . A point ξ of the Grassmannian is expressed by the N -frame of vectors,

$$\xi = [\xi_1, \xi_2, \dots, \xi_N], \quad \text{with} \quad \xi_i = \sum_{j=1}^M a_{ij} e_j \in \mathbb{R}^M,$$

where $\{e_i \mid i = 1, 2, \dots, M\}$ is the standard basis of \mathbb{R}^M , and $(a_{ij}) = A_{(N, M)}$ is the $N \times M$ matrix given in (1.2). Then the Grassmannian $\text{Gr}(N, M)$ can be embedded to the projectivization of the exterior space $\bigwedge^N \mathbb{R}^M$, which is called the Plücker embedding,

$$\begin{aligned} \text{Gr}(N, M) &\hookrightarrow \mathbb{P}(\bigwedge^N \mathbb{R}^M) \\ \xi = [\xi_1, \dots, \xi_N] &\mapsto \xi_1 \wedge \dots \wedge \xi_N \end{aligned}$$

Here the element on $\mathbb{P}(\bigwedge^N \mathbb{R}^M)$ is expressed as

$$\xi_1 \wedge \dots \wedge \xi_N = \sum_{1 \leq i_1 < \dots < i_N \leq M} \xi(i_1, \dots, i_N) e_{i_1} \wedge \dots \wedge e_{i_N},$$

where the coefficients $\xi(i_1, \dots, i_N)$ is an $N \times N$ minors given in Lemma 2.1, which are called the Plücker coordinates.

It is also well known that the Grassmannian can have the cellular decomposition, called the Schubert decomposition [4],

$$(3.1) \quad \text{Gr}(N, M) = \bigsqcup_{1 \leq i_1 < \dots < i_N \leq M} W(i_1, \dots, i_N)$$

where the cells are defined by

$$W(i_1, \dots, i_N) = \left\{ \xi = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ * & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ * & * & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \dots & * \end{pmatrix} \in \overbrace{\mathbb{R}^M \times \dots \times \mathbb{R}^M}^N \right\}$$

$$= \{ \text{the matrix } \xi = A_{(N,M)}^T \text{ in the echelon form} \\ \text{whose pivot ones are at } (i_1, \dots, i_N) \text{ positions} \}$$

Namely an element $\xi = [\xi_1, \dots, \xi_N] \in W(i_1, \dots, i_N)$ is described by

$$\xi \in W(i_1, \dots, i_N) \Leftrightarrow \begin{cases} \text{(i)} & \xi(i_1, \dots, i_N) \neq 0. \\ \text{(ii)} & \xi(j_1, \dots, j_N) = 0 \text{ if } j_n < i_n \text{ for some } n. \end{cases}$$

Each cell $W(i_1, \dots, i_N)$ is called the Schubert cell, and it is convenient to label the cell by the Young diagram $Y = (i_1, \dots, i_k)$ where the number of boxes are given by $\ell_j = i_j - j$ for $j = 1, \dots, N$ (counted from the bottom), which also expresses a partition $(\ell_N, \ell_{N-1}, \dots, \ell_1)$ of the number $|Y| := \sum_{i=1}^N \ell_i$, the *size* of Y . The codimension of the cell $W(i_1, \dots, i_k)$ is then given by the size of Y , i.e.

$$\text{codim } W(i_1, \dots, i_k) = |Y|,$$

and the dimension is given by the number of free variables in the echelon form. Note that the top cell of $\text{Gr}(N, M)$ is labeled by $Y = (1, \dots, N)$, i.e. $|Y| = 0$, and

$$\dim W(1, \dots, N) = \dim \text{Gr}(N, M) = N \times (M - N).$$

The main point here is that each matrix $A_{(N,M)}$ in (1.2) can be identified as a point on a Schubert cell of the Grassmannian. Now we identify an N -soliton solution as a torus orbit of a point on the Grassmannian $\text{Gr}(N, 2N)$, and show that orbits in the different cells give different types of N -soliton solutions.

3.2. N -soliton solution as a $2N$ -dimensional torus orbit. First we note that the τ -function (1.1) in the Wronskian form can be expressed as

$$(3.2) \quad \tau(x, y, t) = \langle v^N, KD(x, y, t)A^t \cdot v^N \rangle,$$

where v^N is the highest weight vector in a fundamental representation of $\text{GL}(2N)$ which is given by an element of $\bigwedge^N \mathbb{R}^{2N}$,

$$v^N = e_1 \wedge e_2 \wedge \dots \wedge e_N,$$

and K, A (A^t represents the transpose of A) are $2N \times 2N$ constant matrices defined by

$$\begin{cases} K = (k_{ij})_{1 \leq i, j \leq 2N}, & \text{with } k_{ij} := (-k_j)^{i-1} \\ A = (a_{ij})_{1 \leq i, j \leq 2N}. \end{cases}$$

The $D(x, y, t)$ is $2N \times 2N$ diagonal matrix associated to an $(\mathbb{R}^*)^{2N}$ -torus action,

$$D(x, y, t) = \text{diag}(e^{\theta_1}, \dots, e^{\theta_{2N}}) \quad \text{with} \quad \theta_j = -k_j x + k_j^2 y - k_j^3 t + \theta_j^0.$$

Note here that $A^t \cdot v^N$ defines the matrix $A_N = A_{(N, 2N)}$ in (1.2), that is,

$$A^t \cdot e_1 \wedge \dots \wedge e_N = A^t e_1 \wedge A^t e_2 \wedge \dots \wedge A^t e_N,$$

which can be identified as A_N^t . The inner product $\langle \cdot, \cdot \rangle$ on $\bigwedge^N \mathbb{R}^{2N}$ is defined by

$$\langle v_1 \wedge \dots \wedge v_j, w_1 \wedge \dots \wedge w_j \rangle = \det[(\langle v_m, w_n \rangle)_{1 \leq m, n \leq j}],$$

where $\langle v_m, w_n \rangle$ is the standard inner product of $v_m, w_n \in \mathbb{R}^{2N}$.

The expression (3.2) implies that the solution $u(x, y, t)$ given by the τ -function (1.1) is a $2N$ -dimensional torus orbit, $(\mathbb{R}^*)^{2N}$ -orbit, of a point on $\text{Gr}(N, 2N)$ marked by A_N . Note also that the Schubert cell $W(i_1, \dots, i_N)$ is invariant under this torus action. The orbits, N -soliton solutions, can be first classified in terms of the Schubert decomposition with the index sets (i_1, \dots, i_N) .

Example 3.1. $\text{Gr}(2, 4)$: The Schubert decomposition of $\text{Gr}(2, 4)$ is given by

$$\text{Gr}(2, 4) = \bigsqcup_{1 \leq i, j \leq 4} W(i, j).$$

There are six cells $W(i, j)$ with $\dim W(i, j) = 7 - (i + j)$.

- i) $W(1, 2)$: This is a top cell of maximum dimension four, and a point on the cell is given by

$$A_2 = \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{pmatrix},$$

which includes 2-soliton solutions of T-type, $A_2^{(T)}$, and P-type, $A_2^{(P)}$.

- ii) $W(1, 3)$: A point on this cell is marked by

$$A_2 = \begin{pmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix}.$$

The 2-soliton solution of O-type with $A_2^{(O)}$ belongs to this cell. Thus O- and P-types belong different cells, even though they show nonresonant interactions.

- iii) $W(1, 4)$: The A_2 matrix has the form,

$$A_2 = \begin{pmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In this case, nonzero minors are given by $\xi(i, 4)$ for $i = 1, 2, 3$. This implies that the exponential term $E_4 = e^{\theta_4}$ becomes a common factor in the

τ -function. Then the solution gives $(2, 1)$ -soliton solution, that is, two incoming solitons and one outgoing soliton in the asymptotics $y \rightarrow \pm\infty$ (see [1]).

iv) $W(2, 3)$: The A_2 matrix is given by

$$A_2 = \begin{pmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix}.$$

Then the exponential term $E_1 = e^{\theta_1}$ is missing in the τ -function, and the solution describes a $(1, 2)$ -soliton solution.

v) $W(2, 4)$: The A_2 matrix is

$$A_2 = \begin{pmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The solution is a 1-soliton solution of $[2, 3]$ -soliton.

vi) $W(3, 4)$: The A_2 matrix is

$$A_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The τ -function is just a product of $E_3 = e^{\theta_3}$ and $E_4 = e^{\theta_4}$, and the solution is trivial, $u = 0$.

Remark 3.2. Since $W(1, 3)$ is a boundary of the top cell $W(1, 2)$, a point on $W(1, 3)$ can be obtained a limit of a sequence of points in $W(1, 2)$. For example, consider the following matrix (representing one parameter family of points in $W(1, 2)$),

$$A_2(\epsilon) = \begin{pmatrix} 1 & 0 & -\epsilon^{-1} & -\epsilon^{-1} \\ 0 & 1 & \epsilon^{-1} + \epsilon & \epsilon^{-1} - \epsilon \end{pmatrix},$$

where the parameter ϵ is a positive number. If $\epsilon < 1$, then the matrix gives 2-soliton solution of T-type (note that all the minors are positive). Then in the limit $\epsilon \rightarrow 0$, the matrix can be interpreted as the O-type matrix $A_2^{(O)}$ in $W(1, 3)$, i.e. (recall here that $\epsilon^{-1}\tau \equiv \tau$ is the Plücker coordinate, i.e. they both give the same solution). It is also obvious that $A_2^{(N)}$ is obtained by a limit of T-type matrix. Thus T-type is generic, while the O-type and P-type are non-generic. Also in the Gallipoli conference in June-July 2004, Pashaev showed that 2-soliton solution of T-type is a degenerate case of 4-soliton solution of nonresonant type (obtained by the Hirota method) [8].

Since N -soliton solution is an orbit on a particular Schubert cell $W(i_1, \dots, i_N)$ which is a boundary of the top cell $W(1, \dots, N)$, it seems to be true that all the N -soliton solutions are obtained by the limits of the τ -function of T-type which lives on the top cell. We will discuss more details in a future communication.

4. CONSTRUCTION OF THE A_N MATRIX

In this section, we give an explicit construction of the matrix A_N for N -soliton solution parametrized by the pair $(\mathbf{n}^+, \mathbf{n}^-)$: The key idea for the construction is based on the classification of 2-soliton solutions, that is, a local structure of the matrix contains one of those types, O-, T- or P-types. With a given pair $(\mathbf{n}^+, \mathbf{n}^-)$, we construct an N -soliton solutions consisting of N line solitons with the labels $[n_j^+, n_j^-]$ for $j = 1, \dots, N$. In order to do this, let us first define:

Definition 4.1. The length of $[n_j^+, n_j^-]$ -soliton, denoted by L_j , is defined by

$$L_j := \min \{ n_j^- - n_j^+, 2N - n_j^- + n_j^+ \}.$$

Note that $1 \leq L_j \leq N$ for all $j = 1, \dots, N$.

In the cases of 2-soliton solutions, we have

- i) for O-type, $L_1 = L_2 = 1$,
- ii) for T-type, $L_1 = L_2 = 2$,
- iii) for P-type, $L_1 = L_2 = 1$.

Then we note that the length of the line soliton gives the condition on the minors. In those cases, $\xi(n_j^+, n_j^-) = 0$ if and only if $L_j < 2$.

For N -soliton solutions with $[n_j^+, n_j^-]$ -solitons, we impose that the τ -functions satisfy the following condition which plays a crucial role for the construction of A_N :

Definition 4.2. We say that a τ -function satisfies the *N -soliton condition*, if it satisfies the duality (Definition 2.3), and has missing exponential terms (i.e. vanishing minors) if and only if there exists $[n_j^+, n_j^-]$ -soliton whose length is less than N . If $L_j < N$ for some j , we require that

- for $L_j = n_j^- - n_j^+$, then the $N \times N$ minors satisfy

$$\xi(\dots, [n_j^+, n_j^+ + 1, \dots, n_j^-], \dots) = 0,$$

- for $L_j = 2N - n_j^- + n_j^+$, then

$$\xi([1, \dots, n_j^+], \dots, [n_j^-, \dots, 2N]) = 0$$

Here the $L_j + 1$ entries inside the brackets $[\dots]$ are consecutive numbers, and the other entries in the “...” outside the $[\dots]$ can take any numbers.

Recall that in the case of N -soliton solutions for the Toda lattice, every soliton has the length N , and there is no vanishing minor [1]. The N -soliton condition is a natural extension of that in the case of 2-soliton solutions, and it provides a local information of 2-soliton interactions in N -soliton solution which are classified into three types, O-, T- and P-types. Note in particular, with this condition, one identifies all the vanishing minors in the τ -function.

Let us now construct the matrix A_N : The construction has several steps, and it might be better to explain the steps using an explicit example.

Let us consider the case $N = 4$ with $\mathbf{n}^+ = (1, 2, 4, 5)$ and $\mathbf{n}^- = (3, 7, 6, 8)$:

Step I: Since the minors $\xi(1, 2, 4, 5) \neq 0$ and $\xi(3, 6, 7, 8) \neq 0$, one can easily see that A_N matrix has the structure,

$$(4.1) \quad \begin{pmatrix} * & * & * & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * \end{pmatrix},$$

where “*” marks a possible nonzero entry, and in particular, the first element in each row should be nonzero (this is the pivot place). Note here that $\xi(1, 2, 3, j) = 0$ for any j , which is also considered as the 4-soliton condition with the length $L_1 = 2$ of $[1, 3]$ -soliton. Notice that $\xi(i_1, i_2, i_3, i_4) = 0$ for any $4 \leq i_1, \dots, i_4 \leq 8$ are dual to $\xi(1, 2, 3, j) = 0$ for some j .

Step II: In this step, we put the form (4.1) into the RREF. First we normalize the first nonzero entries to be 1 in every rows, and eliminate the (2,4)-entry by the third row and the (3,5)-entry by the forth row. Note that this process does not affect the 0's of the matrix in (4.1). Then we have

$$\begin{pmatrix} 1 & * & * & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & * & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * \end{pmatrix}.$$

Further eliminating “*” in the 2nd columns by the 2nd row, we have

$$(4.2) \quad \begin{pmatrix} 1 & 0 & * & 0 & 0 & \# & \# & \# \\ 0 & 1 & * & 0 & 0 & \# & \# & \# \\ 0 & 0 & 0 & 1 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * \end{pmatrix},$$

where the two rows in the 2×3 submatrix with # entries are parallel (recall $\xi(5, 6, 7, 8) = \xi(4, 6, 7, 8) = 0$). Also note that the signs of nonzero entries are uniquely determined by the requirement of all the minors to be nonnegative, for example, the last three entries in the first row are nonpositive, and the signs of the last three entries of the following rows change alternatively. Up to this point, we did not use the order in \mathbf{n}^- . In another words, the form of the matrix in (4.2) is common for any \mathbf{n}^- with a given \mathbf{n}^+ .

Step III: Now we use the orders in \mathbf{n}^- to determine an explicit entries in the matrix for an 4-soliton solution consisting of four line solitons labeled by $[n_j^+, n_j^-]$, i.e. $[1, 3]$, $[2, 7]$, $[4, 6]$, $[5, 8]$. In this step, we identify all other zero elements (i.e. zero minors) of the matrix A_N using the N -soliton condition (Definition 4.2). The following graph shows the overlapping relations among those solitons and the length of each soliton:

$$(4.3) \quad \begin{array}{cccccccc} \circ & - & \bullet & & & & & : L_1 = 2 \\ & \circ & - & - & - & \bullet & & : L_2 = 3 \\ & & & \circ & - & \bullet & & : L_3 = 2 \\ & & & & \circ & - & - & \bullet : L_4 = 3 \end{array}$$

Now we use the 4-soliton condition. First we notice that from $L_1 = 2 < 4$ we have $\xi(1, 2, 3, j) = 0$ for any j (see Step I). Also from $L_3 = 2$, we have $\xi(4, 5, 6, j) = 0$ for any j . In particular, $\xi(1, 4, 5, 6) = 0$ implies that (2, 6)-entry must be zero, so that (1, 6)-entry also zero. Also $\xi(3, 4, 5, 6) = 0$ gives $\xi(1, 2, 7, 8) = 0$ which is also the

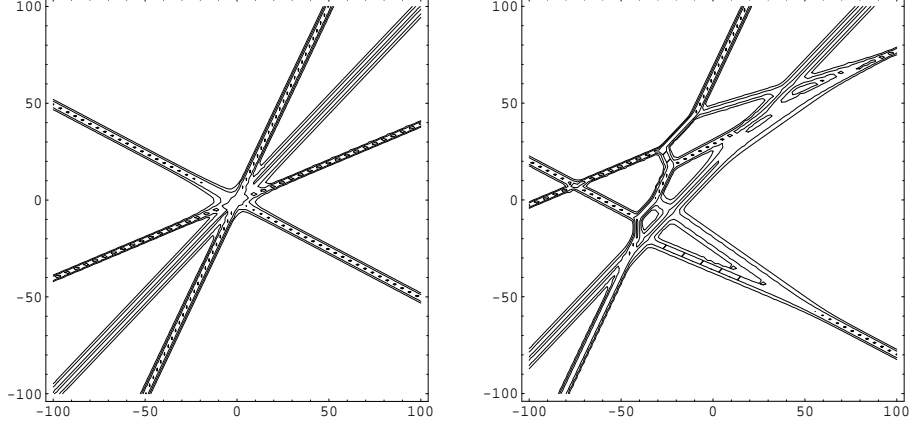


FIGURE 3. 4-soliton solutions labeled by $\mathbf{n}^+ = (1, 2, 4, 6)$ and $\mathbf{n}^- = (3, 7, 6, 8)$. The left figure shows the 4-soliton solution at $t = 0$ and the right one at $t = 10$. The parameters are given by $\theta_j^0 = 0$ for all $j = 1, \dots, 8$, and $(k_1, \dots, k_8) = (-3/2, -1, -1/2, 0, 1/2, 1, 3/2, 2)$. The solution at $t = -10$ is obtained by the reflection $(-x, -y)$ of the right graph.

4-soliton condition with $L_2 = 3$. The $\xi(1, 2, 7, 8) = 0$ implies the 2×2 determinant of the right bottom corner must be zero. We then get the following structure,

$$\begin{pmatrix} 1 & 0 & * & 0 & 0 & 0 & \# & \# \\ 0 & 1 & * & 0 & 0 & 0 & \# & \# \\ 0 & 0 & 0 & 1 & 0 & * & \times & \times \\ 0 & 0 & 0 & 0 & 1 & * & \times & \times \end{pmatrix},$$

where the 2×2 minor with the \times entries is zero. Also from the partial overlaps (i.e. T-type interaction) in $[1, 3]$ with $[2, 7]$, in $[2, 7]$ with $[5, 8]$ and in $[4, 6]$ with $[5, 8]$, we need nonzero entries for those marked by $*$, $\#$ and \times . Then one can easily find an explicit example of the matrix A_4 whose 4×4 minors are all nonnegative,

$$\begin{pmatrix} 1 & 0 & -2 & 0 & 0 & 0 & -3 & -9 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 0 & -1 & -2 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

In Figure 3, we illustrate a 4-soliton solution generated with this matrix. Notice that there are three resonant vertices corresponding to the interactions between $[1, 3]$ and $[2, 7]$, $[2, 7]$ and $[5, 8]$, and $[4, 6]$ and $[5, 8]$ (see the graph 4.3). Other three vertices are two of O-types and one of P-type.

One can show the following Proposition for the general case:

Proposition 4.1. *If there is a pair $[n_j^+, n_j^-]$ with $L_j = n_j^- - n_j^+ < N$, then the $N \times (L_j + 1)$ submatrix (a_{mn}) with $1 \leq m \leq N$ and $n_j^+ \leq n \leq n_j^-$ of the matrix*

A_N has the maximum rank L_j . (Similar thing is also true for the case $L_j = 2N - n_j^- + n_j^+$.)

Proof. Since $L_j < N$, we have $\xi([n_j^+, \dots, n_j^-], *, \dots, *) = 0$ (see the N -soliton condition). Here the entries marked with $*$ are arbitrary columns of A_N whose maximum rank is N . If we choose $N - L_j - 1$ independent columns for those entries, then the $\xi = 0$ implies that the submatrix can have at most the rank L_j . \square

Then the following is immediate as a Corollary of this Proposition:

Corollary 4.1. *If a pair $[n_j^+, n_j^-]$ has the minimum length, i.e. $n_j^- = n_j^+ + 1$, then the $N \times 2$ submatrix with the columns (a_{mn}) with $1 \leq m \leq N$ and $n = n_j^+, n_j^-$ of the matrix should be in the form,*

$$\begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}, \quad \text{where 1's are in the } j\text{th row.}$$

Proof. From Proposition 4.1, the rank of the submatrix is one. This implies that the rows of the submatrix are all proportional. Since the entry (j, n_j^+) is the pivot 1, all the other entries of this column should be zero in the RREF. This gives the result. \square

This Corollary shows that the line soliton marked by $[n_j^+, n_j^-]$ of length one intersects with all other solitons without resonance, that is, the interactions are either O-type or P-type. Notice that in this case $[n_j^+, n_j^-]$ cannot have partial overlap with other pairs, and this can be seen in the matrix A_N . Proposition 4.1 provides the interaction structure of N -soliton solution for O-type and P-type.

We now show that the Young diagrams Y^+ and Y^- provide some information on the interaction pattern of N -soliton solution. If we ignore the phase shifts and resonances among the solitons, N -soliton solution has $\frac{N(N-1)}{2}$ vertices in a generic situation. Then one can find the number of vertices having a particular type of interactions.

Theorem 4.3. *The N -soliton solution labeled by the pair of Young diagrams (Y^+, Y^-) has*

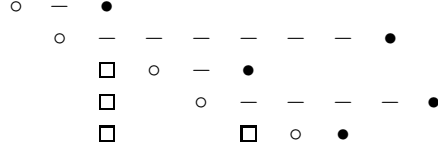
- $|Y^+|$ number of vertices of O-type, and
- $|Y^-|$ number of vertices of P-type.

The number of resonant vertices, T-type, is then given by

$$\frac{N(N-1)}{2} - |Y^+| - |Y^-|.$$

Proof. The Young diagram Y^+ can be obtained directly from the matrix A_N : For example, consider $\mathbf{n}^+ = (1, 2, 4, 5, 7)$ and $\mathbf{n}^- = (3, 9, 6, 10, 8)$. Then the graph

showing the overlaps among $[n_j^+, n_j^-]$ is given by



where the boxes under the non-pivots (\bullet 's) give Y^+ , which is given by $(1, 1, 2)$ representing the number of boxes in the row from the bottom (Y^+ shape is just up-side-down picture of one given in the graph). It is then obvious that each \square provides O-type interaction, i.e. two solitons having no overlap. For example, $[1, 3]$ has no overlap with $[4, 6]$, $[5, 10]$ and $[7, 8]$. This proves that $|Y^+|$ gives the number of vertices of O-type.

The diagram Y^- gives the number of *reverse* in the sequence (n_1^-, \dots, n_N^-) . Again consider the example above. The number $n_2^- = 9$ has two reverses, 6 and 8. Then from the graph above, this implies that $[2, 9]$ has complete overlap with $[4, 6]$ and $[7, 8]$, i.e. $[2, 9]$ -soliton has P-type interaction with those solitons. Now it is obvious that $|Y^-|$ gives the number of vertices of P-type. \square

Before ending this section, we give a complete list of the 3-soliton solutions with explicit examples of A_3 matrices:

Case a: $\mathbf{n}^+ = (1, 2, 3)$: Then we have $m_{\mathbf{n}^+} = 1 \cdot 2 \cdot 3 = 6$ different choice of \mathbf{n}^- . that is, six different 3-soliton solutions. Since the Young diagram Y^+ has no box, $Y^+ = \emptyset$, there is no O-type vertex in those solutions.

- a1. $\mathbf{n}^- = (4, 5, 6)$: This is the case of Toda lattice [1]. The pair of Young diagrams is (\emptyset, \emptyset) , that is, all the vertices are T-type. Example of the matrix A_3 is given by the following matrix. We also show the overlapping graph.

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 3 & 6 \\ 0 & 1 & 0 & -1 & -2 & -3 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad \begin{array}{cccc} \circ & - & - & \bullet \\ & \circ & - & - & \bullet \\ & & \circ & - & - & \bullet \end{array}$$

- a2. $\mathbf{n}^- = (4, 6, 5)$: This has two T-type vertices and one P-type vertex, i.e. $Y^- = \square$. In particular, the P-type interaction between $[2, 6]$ - and $[3, 5]$ -solitons implies that the $(3, 6)$ entry should be zero (because of $\xi(1, 2, 6) = 0$). Then we have an example

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 2 & 4 \\ 0 & 1 & 0 & -1 & -2 & -3 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}, \quad \begin{array}{cccc} \circ & - & - & \bullet \\ & \circ & - & - & \bullet \\ & & \circ & - & \bullet \end{array}$$

Note here that the duality of $\xi(1, 2, 6) = 0$ and $\xi(3, 4, 5) = 0$. Both vanishing minors are due to the 3-soliton condition for $L_2 = L_3 = 2$.

- a3. $\mathbf{n}^- = (5, 4, 6)$: This has again two T-types vertices and one P-type vertex (notice that the Young diagram Y^- is the same as the case 2a). An example of A_3 is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 & -2 & -2 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad \begin{array}{cccc} \circ & - & - & - & \bullet \\ & \circ & - & \bullet & \\ & & \circ & - & - & \bullet \end{array}$$

Again notice the 3-soliton conditions for $L_1 = L_2 = 2$ which give $\xi(1, 5, 6) = 0$ and its dual $\xi(2, 3, 4) = 0$. The $\xi(2, 3, 4)$ then gives (1, 4)-entry to be zero.

a4. $\mathbf{n}^- = (5, 6, 4)$: There are two P-type and one T-type vertices. We have

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & -2 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}, \quad \begin{array}{cccccc} \circ & - & - & - & \bullet & \\ & \circ & - & - & - & \bullet \\ & & \circ & \bullet & & \end{array}$$

Note here that [3, 4]-soliton has P-type interaction with other two solitons, which gives the last row to be $(0, 0, 1, 1, 0, 0)$ and 3rd and 4th columns to be $e_3 \in \mathbb{R}^3$ (see Corollary 4.1).

a5. $\mathbf{n}^- = (6, 4, 5)$: This case is similar to the previous one, i.e. two P-types and one T-type. A_3 is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & -2 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}, \quad \begin{array}{cccccc} \circ & - & - & - & - & \bullet \\ & \circ & - & \bullet & & \\ & & \circ & - & \bullet & \end{array}$$

a6. $\mathbf{n}^- = (6, 5, 4)$: All the vertices are of P-type, and the matrix A_3 can be written by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}, \quad \begin{array}{cccccc} \circ & - & - & - & - & \bullet \\ & \circ & - & - & \bullet & \\ & & \circ & \bullet & & \end{array}$$

The Young diagram Y^- is maximum with three boxes.

In those examples, one should note that the pattern of the zeros in the last three columns (non-pivot columns) presents the shape of the corresponding Young diagram Y^- .

Case b: $\mathbf{n}^+ = (1, 2, 4)$: There are $m_{\mathbf{n}^+} = 1 \cdot 2 \cdot 2 = 4$ different types of 3-soliton solutions. Now the Y^+ has one box, there is one O-type interaction in those solutions.

b1. $\mathbf{n}^- = (3, 5, 6)$: There are two T-type vertices and one O-type. An example of A_3 matrix is

$$\begin{pmatrix} 1 & 0 & -1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}, \quad \begin{array}{cccccc} \circ & - & \bullet & & & \\ & \circ & - & - & \bullet & \\ & & \circ & - & \bullet & \end{array}$$

Notice here the duality in $\xi(1, 2, 3) = 0$ and $\xi(4, 5, 6) = 0$.

b2. $\mathbf{n}^- = (3, 6, 5)$: The Y^- has one box, and the vertices are all different types, O-, N- and T-types. The matrix A_3 can be

$$\begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}, \quad \begin{array}{cccccc} \circ & - & \bullet & & & \\ & \circ & - & - & - & \bullet \\ & & \circ & \bullet & & \end{array}$$

b3. $\mathbf{n}^- = (5, 3, 6)$: This is similar to the previous case of b2. A_3 can be

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}, \quad \begin{array}{cccccc} \circ & - & - & - & \bullet & \\ & \circ & \bullet & & & \\ & & \circ & - & \bullet & \end{array}$$

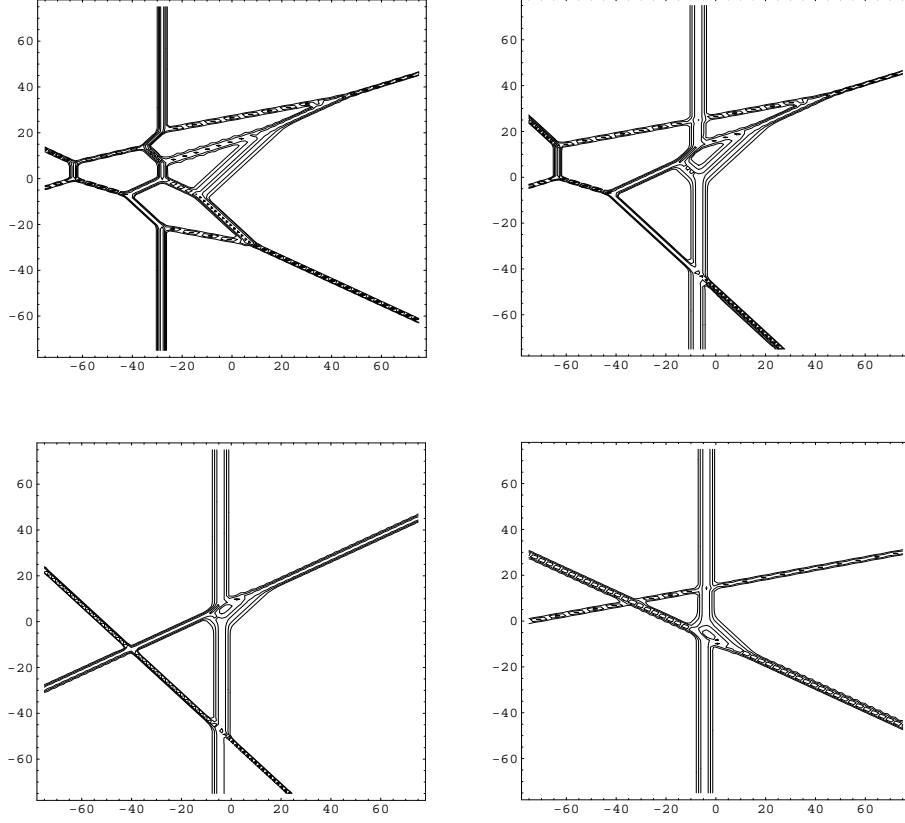


FIGURE 4. 3-soliton solutions. All the cases assume $\theta_j^0 = 0$ for all $j = 1, \dots, 6$. The top left figure shows the solution in the case a1 with the parameters $(k_1, \dots, k_6) = (-3, -2, 0, 1, 2, 3)$ and $t = 7$. The top right one shows the case a3 with $(-3, -1, 0, 1, 2, 3)$ and $t = 7$. The one in the bottom left shows the case a5 with $(-4, -1, 0, 1, 2, 3)$ and $t = 4$, and the one in the bottom right shows the case c1 with $(-2, -1, 0, 1, 2, 3)$ and $t = 4$.

- b4. $\mathbf{n}^- = (6, 3, 5)$: The Y^- has two boxes, and there are two P-type vertices and one O-type. The matrix A_3 is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}, \quad \begin{array}{cccccc} \circ & - & - & - & - & \bullet \\ & \circ & \bullet & & & \\ & & & \circ & \bullet & \end{array}$$

Case c: $\mathbf{n}^+ = (1, 2, 5)$: There are $m_{\mathbf{n}^+} = 1 \cdot 2 \cdot 1 = 2$ different 3-soliton solutions. Since the Y^+ has two horizontal boxes, there are two O-type vertices in those solutions.

- c1. $\mathbf{n}^- = (3, 4, 6)$: The Y^- has no box, and the 3-soliton solution has one T-type interaction with two O-types. An matrix A_3 can be

$$\begin{pmatrix} 1 & 0 & -1 & -2 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad \begin{array}{cccccc} & \circ & - & \bullet & & \\ & & \circ & - & \bullet & \\ & & & & \circ & \bullet \end{array}$$

- c2. $\mathbf{n}^- = (4, 3, 6)$: The Y^- has one box, and the solution has one P-type interaction with two O-types. The matrix A_3 is given by

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad \begin{array}{cccccc} & \circ & - & - & \bullet & \\ & & \circ & \bullet & & \\ & & & & \circ & \bullet \end{array}$$

Case d: $\mathbf{n}^+ = (1, 3, 4)$: The Y^+ has two boxes, and there are $m_{\mathbf{n}^+} = 1 \cdot 1 \cdot 2 = 2$ different 3-soliton solutions which have two O-type interactions.

- d1. $\mathbf{n}^- = (2, 5, 6)$: Since $Y^- = \emptyset$, there is one T-type intersection with two O-types. An example of the matrix A_3 is

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}, \quad \begin{array}{cccccc} & \circ & \bullet & & & \\ & & & \circ & - & \bullet \\ & & & & \circ & - \bullet \end{array}$$

- d2. $\mathbf{n}^- = (2, 6, 5)$: Now Y^- has one box, and the solution has one P-type interaction with two O-types. The matrix A_3 is given by

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}, \quad \begin{array}{cccccc} & \circ & \bullet & & & \\ & & & \circ & - & - \bullet \\ & & & & \circ & \bullet \end{array}$$

Case e: $\mathbf{n}^+ = (1, 3, 5)$: This is the ordinary 3-soliton solution with the matrix,

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad \begin{array}{cccccc} & \circ & \bullet & & & \\ & & & \circ & \bullet & \\ & & & & \circ & \bullet \end{array}$$

Some of the 3-soliton solutions are illustrated in Figure 4.

5. N -SOLITON SOLUTIONS OF THE KdV EQUATION

Here we consider N -soliton solutions of the KdV equation, and show that they cannot have a resonant interaction. N -soliton solutions of the KdV equation can be obtained by the constraint $\partial u / \partial y = 0$ in the KP equation. Since line solitons are given by $[n_j^+, n_j^-]$ -solitons for $j = 1, \dots, N$, the constraint implies that all the solitons are parallel to the y -axis, i.e.

$$c_j = k_{n_j^+} + k_{n_j^-} = 0 \quad \text{for all } j = 1, \dots, N.$$

Now from the ordering $k_1 < \dots < k_{2N}$, we assume k_j 's to satisfy

$$k_1 < k_2 < \dots < k_N < 0, \quad k_{N+j} = -k_{N-j+1} \quad \text{for } j = 1, \dots, N.$$

Then we take the set $(\mathbf{n}^+, \mathbf{n}^-)$ as

$$n_j^+ = j, \quad n_j^- = 2N - j + 1 \quad \text{for } j = 1, \dots, N,$$

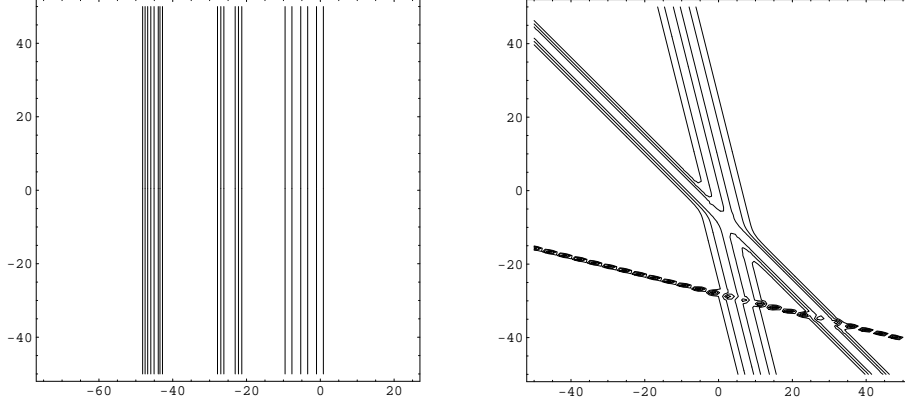


FIGURE 5. 3-soliton solutions of the KdV equation labeled by $\mathbf{n}^+ = (1, 2, 3)$ and $\mathbf{n}^- = (6, 5, 4)$. The left figure shows the 3-soliton solution in x - y plane at $t = 25$ with $\theta_j = 0$, $\forall j$. The right one shows the solution in the x - t plane with different θ_j 's. The k_j 's are given by $(k_1, \dots, k_6) = (-4/3, -1, -1/2, 1/2, 1, 4/3)$.

which leads to $c_j = 0$ for all $j = 1, \dots, N$. In terms of the matrix A_N , this corresponds to

$$A_N = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & * \\ 0 & 0 & \cdots & 0 & 0 & \cdots & * & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & * & \cdots & 0 & 0 \end{pmatrix}.$$

This matrix indicates that any pair of line solitons are of P-type, that is, all the interactions are nonresonant. Each line soliton has the following form,

$$u(x, t) = 2k_j^2 \operatorname{sech}^2 \theta_j, \quad \text{with} \quad \theta_j = -k_j x - k_j^3 t + \theta_j^0.$$

Thus each soliton has the velocity $dx/dt = -k_j^2$. We illustrate a 3-soliton solution of the KdV equation in Figure 5.

Acknowledgements. The author would like to thank G. Biondini for valuable discussions and a Mathematica program for the calculation of the τ -functions. He also thanks to O. Pashaev for showing him the reference [8] before publication and for valuable discussions on AKNS reductions of the KP equation. This research is partially supported by the NSF grant DMS0404931.

REFERENCES

- [1] G. Biondini and Y. Kodama, On a family of solutions of the Kadomtsev-Petviashvili equation which also satisfy the Toda lattice hierarchy, *J. Phys. A: Math. Gen.* **36** (2003) 10519-36.
- [2] N. C. Freeman and J. J. C. Nimmo, Soliton-solutions of the Korteweg-deVries and Kadomtsev-Petviashvili equations: the Wronskian technique *Phys. Lett.* **95A** (1983) 1-3.
- [3] F. R. Gantmacher F R *Theory of Matrices* Vol. 2, (Chelsea, New York, 1959)
- [4] P. Griffiths and J. Harris, *Principles of algebraic geometry* (Wiley, New York 1978).

- [5] R. Hirota, Y. Ohta and J. Satsuma, Wronskian structures of solitons for soliton equations
Prog. Theor. Phys. Suppl. **94** (1988) 59-72.
- [6] E. Infeld and G. Rowlands, *Nonlinear waves, solitons and chaos* Cambridge University Press,
Cambridge 1990, 205-212.
- [7] B. B. Kadomtsev and V. I. Petviashvili, On the stability of solitary waves in weakly dispersive
media, *Sov. Phys. Dok.* **15** (1970) 539-541.
- [8] O. K. Pashaev and M. L. Yerbag-Francisco, Degenerate four virtual soliton resonance for
KP-II, Preprint (June 2004).
- [9] M. Sato, Soliton equations as dynamical systems on an infinite dimensional Grassmann man-
ifolds, *RIMS Kokyuroku*, **439** (1981) 30-46.

DEPARTMENT OF MATHEMATICS, OHIO STATE UNIVERSITY, COLUMBUS, OH 43210

E-mail address: kodama@math.ohio-state.edu